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CONTINUOUS ARCHES ON ELASTIC PIERS

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CONTINUOUS ARCHES ON ELASTIC PIERS

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SYNOPSIS

A procedure that involves successive corrections is presented for determining moments and thrusts in continuous arches. Exact influence lines are obtained by making an analysis for only one position of a unit load in each span. The effects of pier dimensions on thrusts and moments are studied.

INTRODUCTION

A structural analysis of continuous arches on slender piers can be made through the use of classical structural mechanics. Various specific techniques are available, but their application to this problem is in general rather involved and cumbersome.

An analysis can also be made through use of more modern numerical methods that involve successive corrections to a set of assumed values. Such methods are based on the philosophy of moment distribution.[3]

A simple and direct numerical method that involves alternate distribution of thrusts and moments was presented by Larson.[4] However, the convergence is quite slow. To speed the convergence, a procedure has been used in which the thrusts are distributed at a so-called neutral point of each joint.[5] A faster convergence is now obtained at the expense of simplicity. Others, such as Hrennikoff,[6] Morgan,[7] and Maugh,[8] have also introduced variations of the general method.

In this paper an analytical procedure is presented that involves aspects of classical and modern theory familiar to structural engineers. As a matter

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1. Prof. of Structural Eng. and Chairman, Dept. of Civ. Eng., New York Univ., New York, N. Y.
2. Detailer, Howard, Needles, Tammen and Bergendoff, Cons. Engrs., Kansas City, Mo.
3. "Analysis of Continuous Frames by Distributing Fixed-End Moments," by Hardy Cross, Transactions, ASCE, Vol. 97, 1932, pp. 1-10.
4. Ibid. Discussion by Donald E. Larson, pp. 127-133.
5. "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, John Wiley & Sons, Inc., New York, 1932, pp. 316-326.
6. "Analysis of Multiple Arches," by A. Hrennikoff, Transactions, ASCE, Vol. 101, 1936, p. 388.
7. "An Exact Method of Analysis of Continuing Parabolic Arches," by V. A. Morgan, Conc. and Const. Eng., London, Vol. 47, No. 11, Nov. 1952, p. 343.
8. "Statically Indeterminate Structures," by L. C. Maugh, John Wiley & Sons, Inc., New York, 1946, pp. 230-242.

of fact, the general pattern of computations is one widely used by engineers to analyze rigid frames subject to sidesway. A scheme is also presented for obtaining exact influence lines by making only one analysis for each span of the structure. Finally the results of some studies are presented to illustrate the effect of pier dimensions on thrust.

Analytical Procedure

Essentially the analytical procedure involves the distribution of fixed-end moments to joints restrained against translation, and the subsequent addition of corrections because of the actual translation of the joints. This is the general procedure widely used by engineers in the analysis of rectangular bents.

The continuous arch shown in Fig. 1 has been used elsewhere[5] to illustrate an analytical procedure involving successive distribution of moments and thrusts about the so-called neutral points of the joints. It will now be used to illustrate the procedure proposed herein.

The arches are all of the same type. According to Whitney's[9] classification, N = 0.20 and m = 0.40. The shape coefficient N represents the ratio of the drop of the arch axis at the quarter point to the rise of the arch. The form coefficient m represents the ratio of $\frac{\Delta s}{I}$ at the springing to $\frac{\Delta s}{I}$ at the crown for the same Δx.

As in ordinary moment distribution, it is first necessary to obtain relative values of the moment stiffnesses and the carry-over factors. Because the piers are prismatic, their stiffness is equal to 4 EI_p/L_p, in which I_p represents the moment of inertia of the pier cross section and L_p represents the height of the pier. The corresponding carry-over factor is 0.5. Values for non-prismatic piers can be obtained from available tables and charts.

The moment stiffness for each arch is 19.62 EI_a/L_a, in which I_a represents the moment of inertia of the rib cross section at the crown and L_a represents the span of the arch. The carry-over factor for each arch is 0.444. See the Appendix for determination of these values, as well as for tabulated values for arches of constant cross section.

If I_a is considered proportional to the cube of the crown depth, and if the relative stiffness of arch AB is taken as 1.000, then the relative stiffnesses of the other members can be obtained as follows:

<u>Arches</u>

$$\text{BC:} \quad 1.000\left(\frac{50}{75}\right)\left(\frac{22.25}{16}\right)^3 = 1.793$$

$$\text{CD:} \quad 1.000\left(\frac{50}{100}\right)\left(\frac{28.75}{16}\right)^3 = 2.901$$

$$\text{DE:} \quad 1.000\left(\frac{50}{60}\right)\left(\frac{17.50}{16}\right)^3 = 1.090$$

9. "Design of Symmetrical Concrete Arches," by Charles S. Whitney, Transactions, ASCE, Vol. 88, 1932, p. 931.

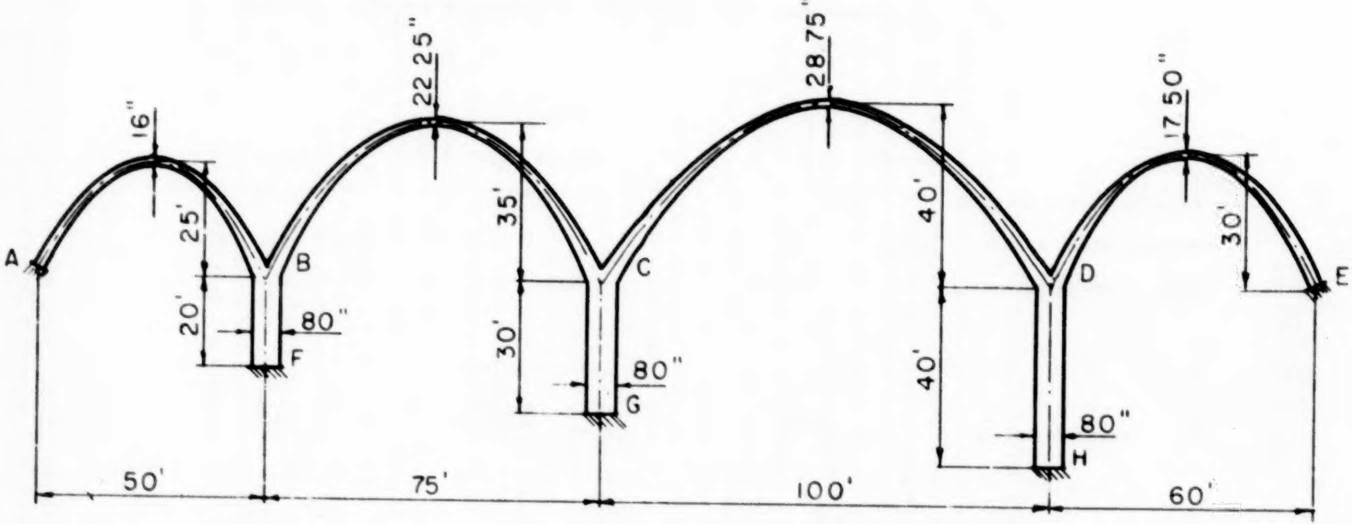


Figure 1. Continuous Arch Series.

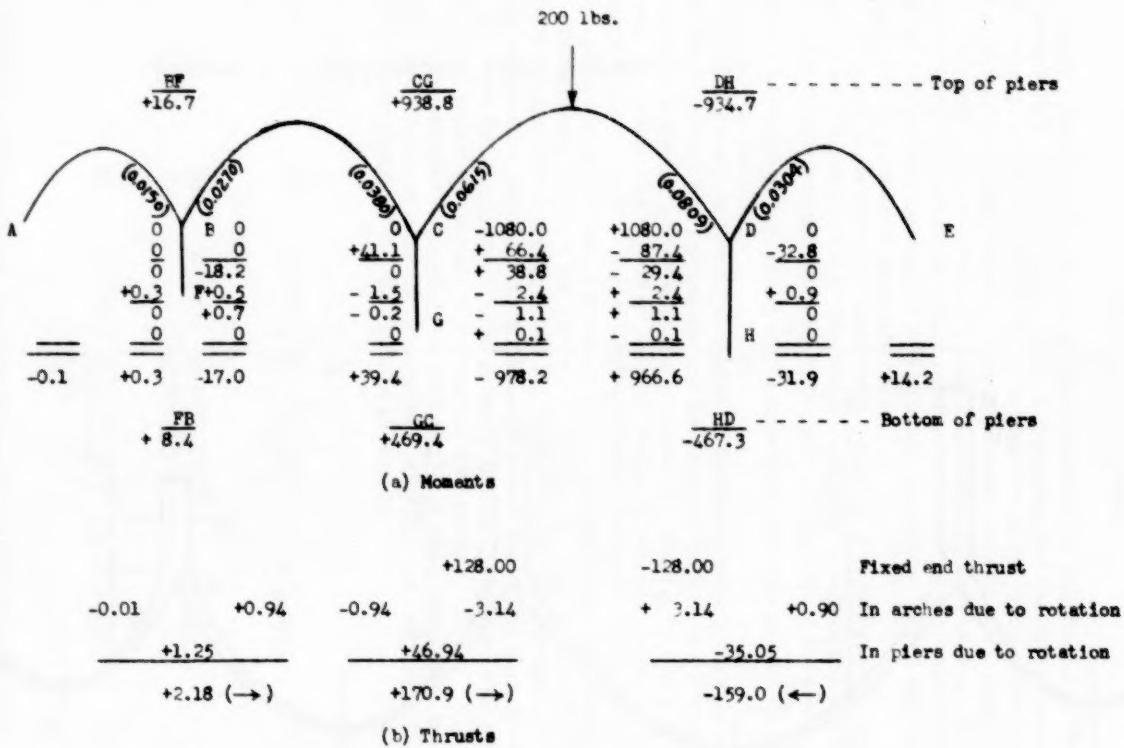


Figure 2. Moments and Thrusts--Displacement of Joints Prevented

Piers

$$\text{BF:} \quad \frac{4}{19.62}\left(\frac{50}{20}\right)\left(\frac{80}{16}\right)^3 = 63.71$$

$$\text{CG:} \quad \frac{20}{30}\left(63.71\right) = 42.47$$

$$\text{DH:} \quad \frac{20}{40}\left(63.71\right) = 31.86$$

An unbalanced moment at a joint is distributed in proportion to the stiffnesses of the members framing into the joint. The relative amounts of unbalanced moment taken by each rib are shown in parentheses at each joint. For example, at C the distribution factors are 0.0380 for CB and 0.0615 for CD. Since no intermediate distributions need be made to the pier, it is not necessary to enter its distribution factor of 0.9005.

Fixed-end moments and horizontal thrusts for any particular arch and case of loading can be determined by means of any one of several well-known analytical procedures. Whitney[9] and Robinson[10] give values of fixed-end moment for arches of certain defined characteristics.

The fixed-end moments resulting from the load of 200 lb. placed at the crown of arch CD are, as given in Cross and Morgan[5] from Whitney's curves,[9] - 1080 ft. lbs. at C and + 1080 ft. lbs. at D. Moments tending to turn a joint clockwise are considered positive. The fixed-end moments are distributed in Fig. 2a. No intermediate distribution is made to the pier. The moment at the top of pier CG is 978.2 - 39.4, or 938.8 ft. lbs. One-half of this, or 469.4, is carried over to the bottom of the pier.

The fixed-end horizontal thrust of the loaded span is, again from Cross & Morgan, 128 lbs. Thrust to the right is designated positive and thrust to the left is designated negative. The thrusts in the arches and piers due to rotation of the joints are combined, in Fig. 2b, with the fixed-end thrusts to obtain the thrusts corresponding to the applied loads but with displacement of all joints prevented.

The arch thrust due to rotation of a joint is found by multiplying the change in end moments in a span by 0.5837/h, in which h is the rise of the arch. The expression 0.5837/h represents, for all arches in this example, the thrust due to a unit moment applied, without allowing translation, at one end of the arch when the other end is free to rotate. It has been derived in the Appendix. Also see the Appendix for tabulated values for arches of constant cross section. The thrust in the pier due to rotation of the joint is found as the algebraic sum of the moments at top and bottom divided by the pier height. The direction of the thrust in each instance is determined by inspection.

10. "Influence Lines for the Design of Hingeless Arches," by Rheese Richard Robinson, unpublished M. S. Thesis, Iowa State College Library, Ames, Iowa, 1951.

At joint D:

Fixed-end thrust in span CD	= - 128.0 lbs.
Thrust in CD due to rotation of joint D	
= (1080 - 978.2 + 1080 - 966.6) 0.5837/40	= + 3.1
Thrust in DE due to rotation of joint D	
= (31.9 + 14.2) 0.5837/30	= + 0.9
Thrust in pier DH due to rotation	
= (934.7 + 467.3)/40	= - 35.0
Total unbalanced thrust	= - 159.0 lbs.

The total at each joint is the horizontal force that must be applied externally to prevent translation of the joint, and it is now necessary to correct the analysis for the effect of these forces.

Arbitrary, but equal for convenience, horizontal displacements without rotation are introduced successively at joints B, C, and D, and the corresponding moments are obtained. For the prismatic piers such moments are proportional to $6I_p/L_p^2$, and for the arches of this example they are proportional to $16.53\ I_a/hL_a$. See the Appendix for derivation of the latter expression, as well as for tabulated values for arches of constant cross section.

If due to displacing joint B the moments at the ends of span AB are assumed to be 1000, the corresponding moments for the other members are obtained as follows:

$$\text{BC:}\quad 1000\left(\frac{25}{35}\right)\left(\frac{50}{75}\right)\left(\frac{22.25}{16}\right)^3 = 1280$$

$$\text{CD:}\quad 1000\left(\frac{25}{40}\right)\left(\frac{50}{100}\right)\left(\frac{28.75}{16}\right)^3 = 1813$$

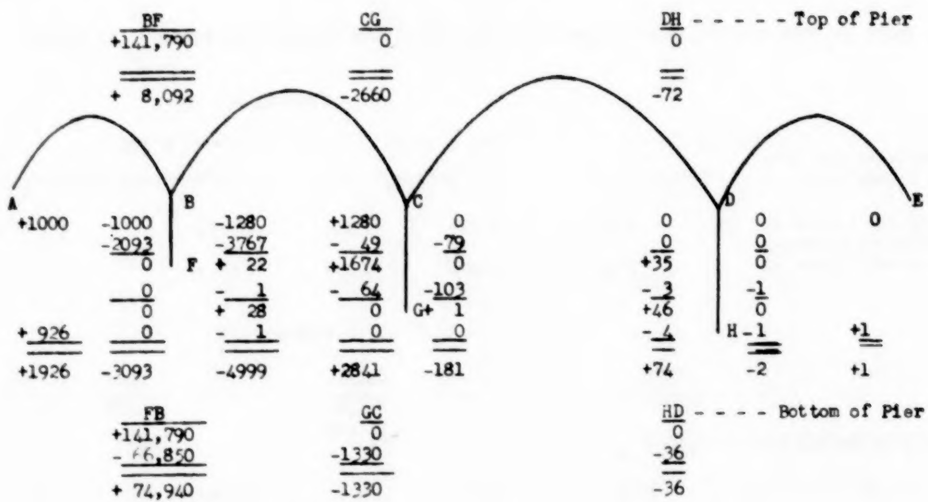
$$\text{DE:}\quad 1000\left(\frac{25}{30}\right)\left(\frac{50}{60}\right)\left(\frac{17.50}{16}\right)^3 = 909$$

$$\text{BF:}\quad 1000\left(\frac{6}{16.53}\right)\left(\frac{80}{16}\right)^3\frac{(25)(50)}{(20)^2} = 141{,}790$$

$$\text{CG:}\quad 141{,}790\left(\frac{20}{30}\right)^2 = 63{,}020$$

$$\text{DH:}\quad 141{,}790\left(\frac{20}{40}\right)^2 = 35{,}450$$

In Figs. 3, 4, and 5 moments and thrusts are obtained due to displacement to the right of joints B, C, and D successively. The moments are distributed in the upper part of each of these figures. No intermediate distributions or carry-overs are made at top and bottom of the piers. For example, at joint B in Fig. 3 (a) the final moment, + 8092, at the top of the pier is obtained as the

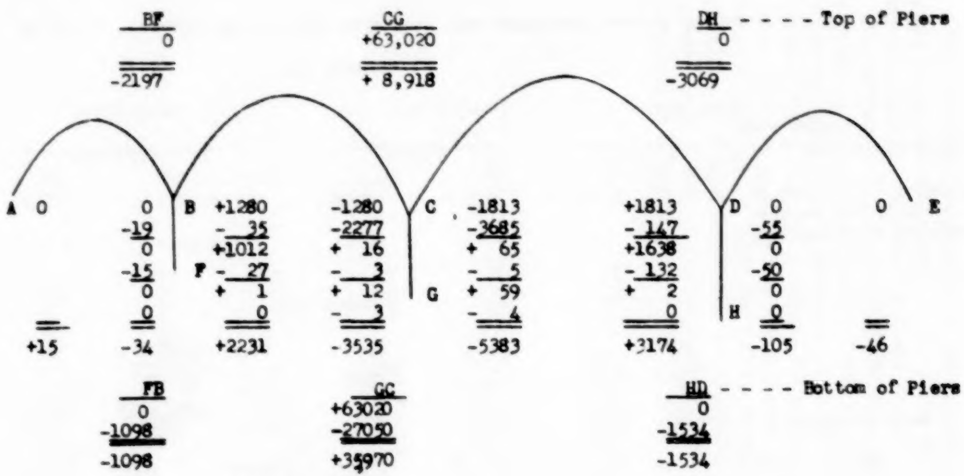


(a) Moments

+51.9	+ 47.6	-47.6				In arches due to translation with no rotation.
+70.5	+ 88.0	-88.0	+3.72	-3.72	+0.06	In arches due to rotation
+4152.0		-133.0		-2.70		In piers due to displacement and rotation.
+4410 (⟶)		-264.9 (⟵)		-6.36 (⟵)		

(b) Thrusts

Figure 3. Moments and Thrusts Resulting from Arbitrary Lateral Displacement of Joint B



(a) Moments

	-47.6	+47.6	+58.9	-58.9		In arches due to translation with no rotation.
+ 1.1	-53.4	+53.4	+71.9	-71.9	+2.9	In arches due to rotation.
- 165.0		+1496.2		-115.1		In piers due to displacement and rotation.
- 264.9 (←)		+1728.0 (→)		-243.0 (←)		

(b) Thrusts

Figure 4. Moments and Thrusts Resulting from Arbitrary Lateral Displacement of Joint C

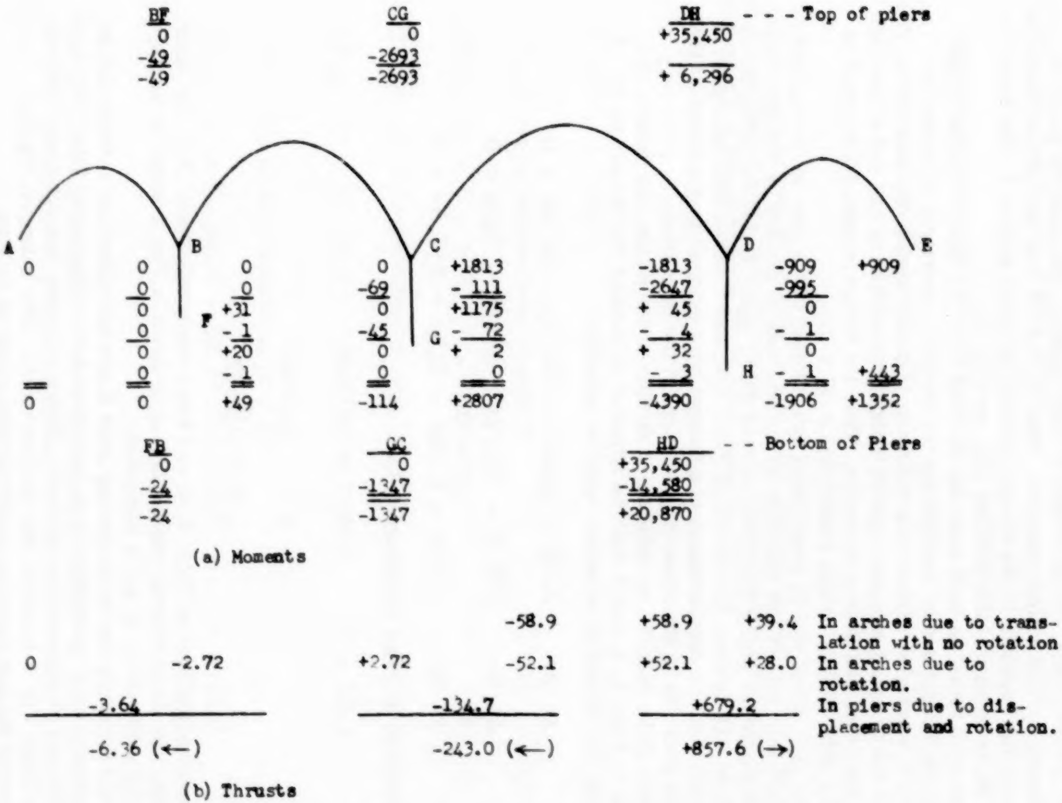


Figure 5. Moments and Thrusts Resulting from Arbitrary Lateral Displacement of Joint D

sum, with opposite sign, of the arch moments. The moment at the bottom of the pier, + 74,940, is obtained by subtracting from the fixed-end moment one-half of the total moment distributed to the top of the pier, i. e., (141,790 - 8,092) 1/2, or - 66,850.

The determination of thrusts will now be illustrated in connection with Fig. 3. The arch thrust in each span due to displacement only is found by dividing the corresponding fixed-end moment by 0.7704 h. The expression 0.7704 h represents, for all arches in this example, the moment due to a unit thrust applied without rotation. That is, 0.7704 h is the vertical distance, y_c, from the supports to the elastic centroid of these arches.[5] The thrust in span BC is equal to 1280/(0.7704) (35), or 47.6 lbs.

The arch thrust in each span due to rotation, i. e., due to change in moments, is determined by multiplying the change in moments in a span by 0.5837/h, where, as explained previously, this expression represents the thrust due to a unit moment applied without translation at one end of the arch when the other end is free to rotate. Thus the thrust in span BC is equal to (4999 - 1280 + 2481 - 1280) 0.5837/35, or 88.0 lbs.

The pier thrust due to translation and rotation of the pier is determined as the algebraic sum of the moments at top and bottom divided by the pier height. For example, in Fig. 3 the thrust in pier CG is equal to (2660 + 1330)/30, or 133.0 lbs. Of course, inspection of Figs. 3, 4, and 5 shows that the thrust at any joint due to a displacement at a second joint is equal to the thrust at the second joint due to the same displacement at the first joint.

Since there must be no unbalanced thrusts at the joints, the thrusts obtained in Figs. 3, 4, and 5 must be adjusted to balance the thrusts in Fig. 2. That is, the following equations must be satisfied:

$$\text{At B:} \quad 2.18 + 4410x - 264.9y - 6.36z = 0$$

$$\text{At C:} \quad 170.9 - 264.9x + 1728y - 243z = 0$$

$$\text{At D:} \quad -159 - 6.36x - 243y + 857.6z = 0$$

The solution of these equations yields:

$$x = -0.00486$$

$$y = -0.07663$$

$$z = +0.1637$$

The computations in Figs. 3, 4, and 5 are made only once. For any other loading, new computations corresponding to those in Fig. 2 would be made, and new values of x, y, and z computed.

If the moments due to displacing joint B are multiplied by x, those due to displacing joint C multiplied by y, and those due to displacing joint D by z, an adjusted set of moments is obtained. These moments are combined with moments from Fig. 2 to obtain the final moments. For example, the final moment at the left end of span CD is determined as follows:

Moment due to applied loads, but with displacement prevented (Fig. 2) = - 978 ft. lbs.

Moment due to displacement of joint B

(Fig. 3) = (-181) (-0.00486) = + 1

Moment due to displacement of joint C

(Fig. 4) = (-5383) (-0.07663) = + 412

Moment due to displacement of joint D

(Fig. 5) = (+2807) (+0.1637) = + 459

- 106 ft. lbs.

Final thrusts can be obtained in a similar manner by adding to the thrusts in Fig. 2 the adjusted thrusts resulting from displacement of the joints. For example, the thrust in span CD is determined as follows:

Thrust due to applied loads, but with displacement prevented (Fig. 2) =

+ 128.0 - 3.14 = 124.9 lbs.

Thrust due to displacement of joint B

(Fig. 3) = (- 3.72) (- 0.00486) = 0

Thrust due to displacement of joint C

(Fig. 4) = (+ 58.9 + 71.9) (-0.07663) = - 10.1

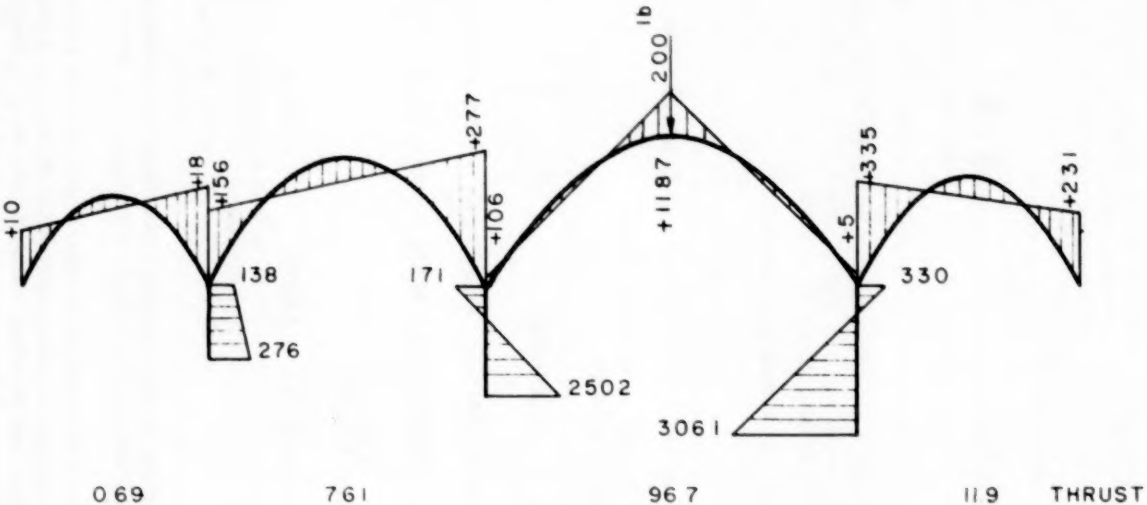
Thrust due to displacement of joint D

(Fig. 5) = (- 58.9 - 52.1) (+ 0.1637) = - 18.1

+ 96.7 lbs.

In the preceding computations for final moment and thrust in the loaded span it is worth observing the unimportance of the translation of a pier of an adjoining span.

Final moments and thrusts for the entire structure are shown in Fig. 6. The shears in the piers are in each instance equal to the difference of the thrusts in the adjoining arches, and the shear multiplied by the pier height should check the sum or difference of the moments at top and bottom of the pier.



Moments are in foot pounds and thrusts in pounds.
+ indicates tension on intrados of arches
Moment diagrams for piers are drawn on tension side.

Figure 6. Moment Diagram.

To study the relative importance of rotation and translation of the pier tops, the two effects can be separated as shown in the following for C in span CD:

1. Change in fixed-end moments:
   Total change in moment at left end of span

CD = - 106 - (- 1080) = 974 ft. lbs.

Change in moment due to translation of pier tops (see Figs. 4 and 5) =

(- 1813) (- 0.07663) + (+ 1813) (+ 0.1637) = 436 ft. lbs.

Change in moment due to rotation of pier tops = 538 ft. lbs.

2. Change in fixed-end thrusts:

Total change in thrust = 96.7 - 128.0 = - 31.3 lbs.

Change in thrust due to translation of pier tops (see Figs. 4 and 5) =

(+ 58.9) (- 0.07663) + (- 58.9) (+ 0.1637) = - 14.1 lbs.

Change in thrust due to rotation of pier tops = - 17.2 lbs.

The results shown in Fig. 6 can be obtained also by distributing the fixed-end thrusts and then correcting for rotation at the joints. The moment distribution procedure is preferred because of the similarity to the rather common use of moment distribution in the analysis of rectangular frames. Furthermore it is simpler to apply the moment distribution procedure to continuous arches with piers hinged at their bases. Thrust distribution would make necessary the application of arbitrary rotations at each of these hinged bases as well as at the tops of the piers.

Moments and thrusts resulting from volume changes or movements of supports can be obtained by distributing the moments due to displacements without rotation.

Influence Lines for Continuous Arches

Influence lines for arches on slender piers differ from those for fixed arches in that allowance must be made for the translation and rotation of the pier tops.

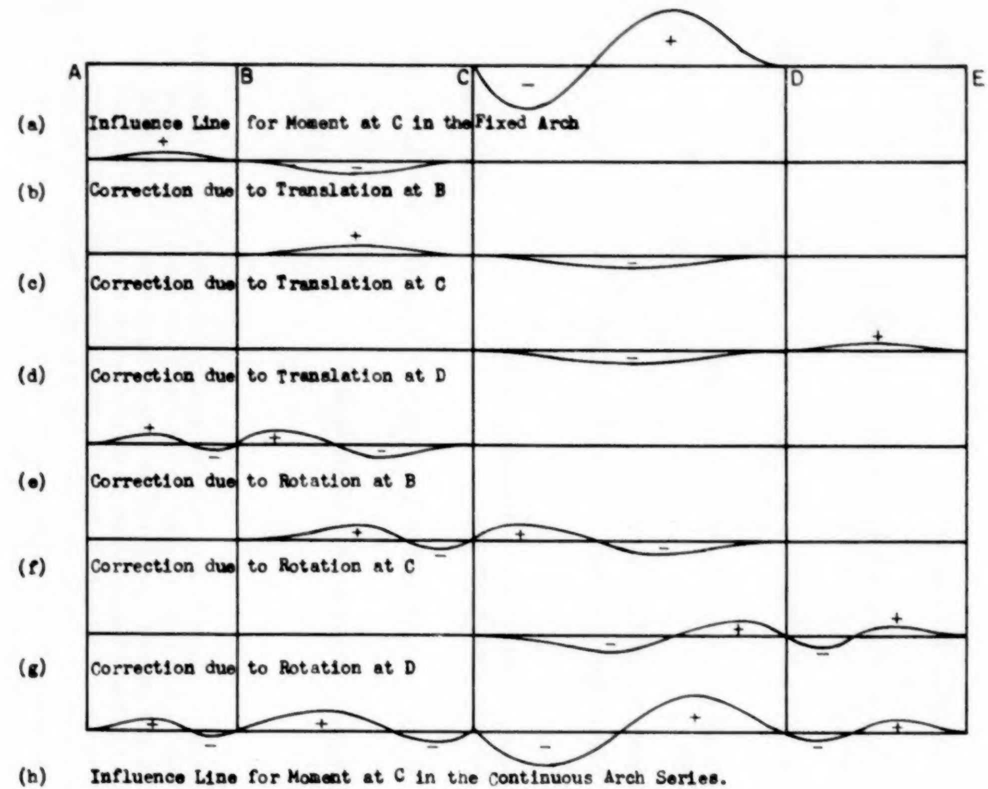
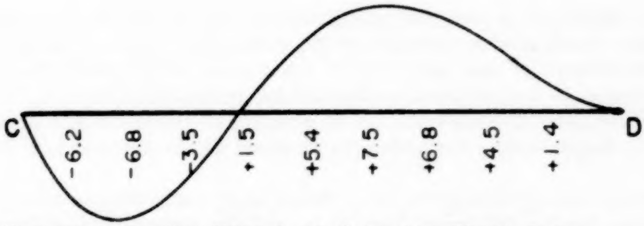
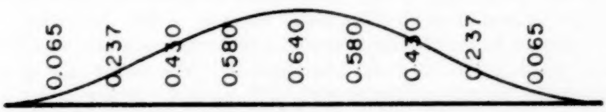


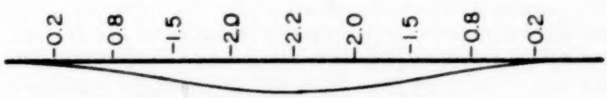
Figure 7. Construction of Influence Line.



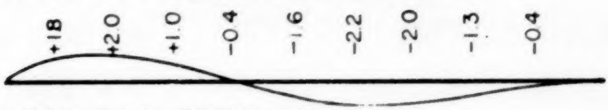
(a) Influence Line for Moment at C in the Fixed Arch.



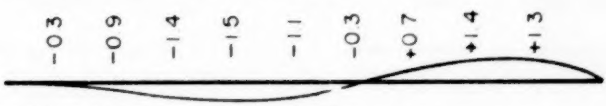
(b) Influence Line for Thrust in the Fixed Arch.



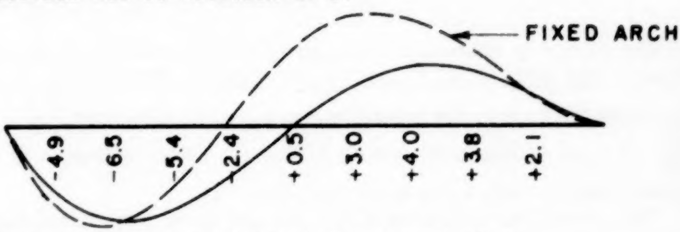
(c) Correction due Translation of Pier Tops.



(d) Correction due to Rotation at C.



(e) Correction due to Rotation at D.



(f) Influence Line (span DC only) for Moment at C.

Figure 8. Detail Construction of Influence Line.

A procedure is introduced herein for obtaining exact influence lines by making an analysis for only one position of a unit load in each span. The influence lines are obtained by superimposing the following corrections on influence lines for moment in a fixed arch:

1. Corrections for translation of piers.
2. Corrections for rotation of piers.

The procedure is outlined graphically in Fig. 7 for the structure in Fig. 1. Moments which produce tension in the outer fibers are considered negative. The corrections in each span due to translation of the piers are directly proportional to the influence ordinates for horizontal thrust in the fixed arch. The corrections in each span due to rotation of the piers are directly proportional to the influence ordinates for moment at the springing of the fixed arch.

The procedure is illustrated in detail, with reference to span CD, in Fig. 8. Ordinates for the influence lines in (a) and (b) were obtained from Whitney's curves.[9] The center ordinate of the curve in (c) represents the change in moment at the springing due to translation of the piers when a unit load is placed at the crown of arch CD. Near the end of the preceding section of this paper the change in fixed-end moment due to translation was found to be 436 ft. lbs. when a 200-lb. load was applied. The value due to a unit load at the crown of CD is then equal to 436/200, or 2.2. Other ordinates to the curve in (c) are determined by ratio from the ordinates in (b).

The center ordinates of the curves in (d) and (e) represent the change in moment at springing C due to rotation of piers C and D respectively when a unit load is placed at the crown of arch CD. The change in moment at C due to rotation of both piers was previously found to be 538 ft. lbs. The change in moment due to rotation of pier C alone can be found by adding only the distributed moments from Fig. 2a to the adjusted distributed moments from Figs. 3a, 4a, and 5a, as follows:

$+ 66.4 - 2.4 + 0.1$	$= + 64.1$ (Fig. 2a)
$- 0.00486\ (- 79 - 103)$	$= + 0.9$ (" 3a)
$- 0.07663\ (- 3685 - 5 - 4)$	$= +282.0$ (" 4a)
$+ 0.1637\ (- 111 - 72)$	$= - 30.0$ (" 5a)
	$+317.0$ ft. lbs.

The change in moment at springing C due to rotation of pier D only is then equal to 538 - 317 = 221 ft. lbs. If, instead of a 200 lb. load, a unit load is placed at the crown, the moments due to rotation become $\frac{317}{200} = 1.6$, and $\frac{221}{200} = 1.1$, as shown in (c) and (d) of Fig. 8. Other ordinates to the curves are determined by ratio from the ordinates in (a).

The correction ordinates in (c), (d), and (e) of Fig. 8 are then added to the influence ordinates in (a) to obtain the influence line in (f). The curve of (a) is shown dashed for comparison.

Effect of Pier Dimensions on Thrusts and Moments in Arch Ribs

In this section results are presented of a series of studies made to show the effect of pier dimensions on changes in fixed-end thrusts and moments. Such studies are of particular interest to the designer.

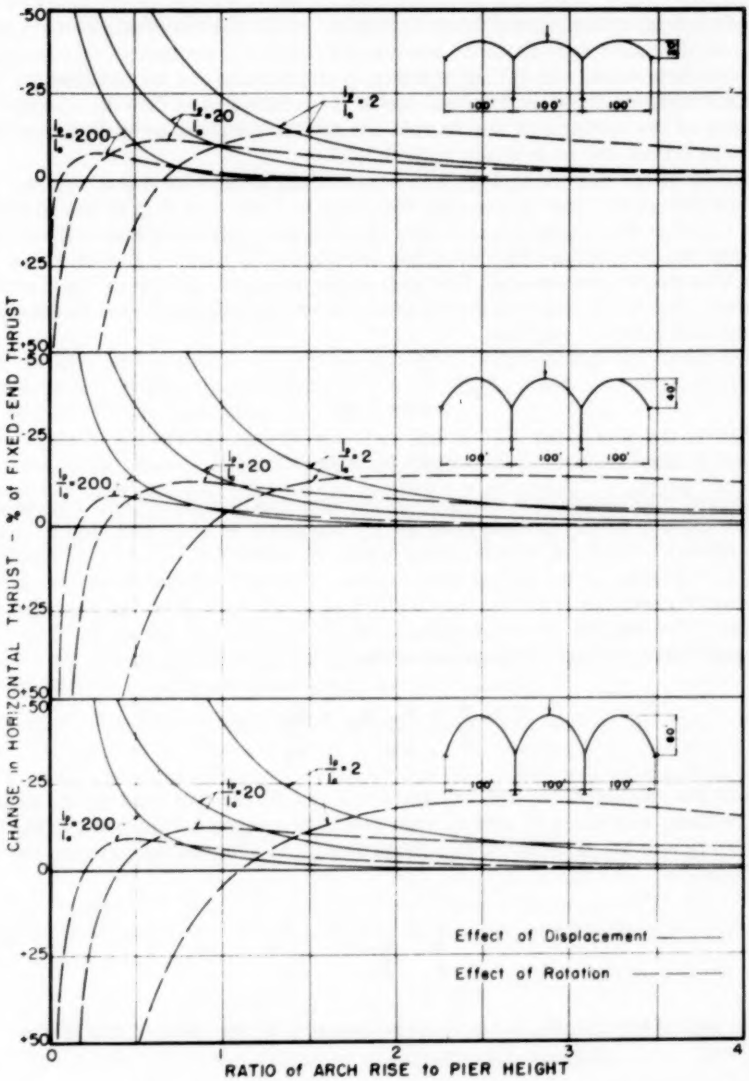


Figure 9. Changes in Thrust due to Displacement and Rotation

For any particular variation in cross section and shape of rib, and for any particular variation in section of piers, the parameters that define the relative proportions of the structure are the ratio of rise of arch to height of pier, h/L_p, the ratio or rise of arch to length of arch, h/L_a, and the ratio of moment of inertia of pier to moment of inertia of arch at crown, I_p/I_a.

Curves are drawn in Fig. 9 to show the percentage changes in fixed-end thrust due to a concentrated load placed at the crown of the center arch of a

symmetrical three-span structure. Three series of arches with different ratios of rise to span are included. Changes in fixed-end thrust resulting from displacement and rotation of the pier tops are plotted separately.

Changes in moments have not been plotted, but they can be obtained readily from the changes in thrust plotted in Fig. 9. Since the arches under consideration are the same type as those analyzed in the first section of this paper, change in moment at the springings due to translation can be obtained by multiplying change in fixed-end thrust due to translation by 0.7704 h. Change in moments at the springings due to rotation can be obtained by multiplying the change in thrust due to rotation by h/0.5837.

A study of the curves in Fig. 9 indicates that, except for relatively very long and thin piers, the percentage decrease in fixed-end thrust due to translation and that due to rotation are fairly constant. For such cases it can be seen that the percentage decrease due to rotation is roughly two or more times that due to translation. For extremely long and thin piers the horizontal thrust decreases sharply due to translation but increases due to rotation, although not quite as rapidly.

APPENDIX

Moment Stiffnesses and Carry-Over Factors

Moment stiffnesses and carry-over factors can be determined conveniently through the use of the column analogy.[11] Moments at each end, due to a unit angle change at one end of a member when the other end is fixed, are computed as stresses in the analogous column. The desired end moments are the "stresses" resulting from the application of a unit load P on the analogous column at the section corresponding to the rotated end of the arch. The "stresses" are computed by means of the familiar expression

$$f = \frac{P}{A} + \frac{M_x}{I_x} y_c + \frac{M_y}{I_y} x_c \tag{1}$$

Since the moments M_x and M_y are those due to the unit load on the analogous column, and since y_c and x_c represent the centroidal distances, M_x is equal to y_c and M_y is equal to x_c. Equation 1 can then be re-written, for a unit "load,"

$$f = \frac{1}{A} + \frac{y^2_c}{I_x} + \frac{x^2_c}{I_y} \tag{2a}$$

for the end of the column section corresponding to the rotated end of the arch, and

$$f = \frac{1}{A} + \frac{y^2_c}{I_x} - \frac{x^2_c}{I_y} \tag{2b}$$

11. "The Column Analogy," by Hardy Cross, Bulletin No. 215, Eng. Experiment Station, University of Illinois, Urbana, Illinois, 1930.

for the opposite end. The "stress" obtained from Eq. 2a is equal to the moment stiffness, and the "stress" obtained from Eq. 2b is equal to the carry-over moment.

Attention is called to the fact that the subscripts x and y are interchanged with those used by Professor Cross, the more usual practice being followed here.

For any particular arch the individual terms of Eqs. 2 can be evaluated, and moment stiffness and carry-over factor obtained. The values for the arches in Fig. 1 are the following:[5]

$$\frac{1}{A} = 1.43 \frac{EI_a}{L_a}$$

$$\frac{y^2{}_c}{I_x} = (0.7704\ h)^2 \left(\frac{21.46}{h^2}\right) \frac{EI_a}{L_a} = 12.74 \frac{EI_a}{L_a}$$

$$\frac{x^2{}_c}{I_y} = 5.45 \frac{EI_a}{L_a}$$

The moment stiffness for all arches in Fig. 1 is (from Eq. 2a) 19.62 EI_a/L_a. The carry-over moment (from Eq. 2b) is 8.72 EI_a/L_a, and the carry-over factor is 8.72/19.62, or 0.444.

The moment stiffnesses and carry-over factors for the piers, all of which are prismatic members, are equal to 4 EI_p/L_p and 0.5 respectively.

Moment stiffnesses and carry-over factors for parabolic, circular, and semi-elliptic arches of constant section have been evaluated on the basis of previous work.[10] Values for several ratios of rise to span are given in Fig. 11.

Change in Thrusts due to Joint Rotation

The thrust due to a unit moment applied at one end of an arch with the other end free to rotate may be determined as follows:

1. Compute the thrust for a unit moment applied at one end with the other end fixed. This can be determined by dividing the thrust, y_c/I_x, due to a unit rotation by the moment stiffness. That is, the thrust due to a unit moment, far end fixed, is

$$(0.7704h) \frac{21.46}{h^2} \frac{EI_a}{L_a} \div 19.62 \frac{EI_a}{L_a} = \frac{0.8427}{h}$$

2. Now release the fixed end and distribute moments as shown in Fig. 10. The change in moments produces a change in thrusts, so that the thrust due to a unit moment far end free to rotate, is

$$\frac{0.4685}{h} \div 0.8026 = \frac{0.5837}{h}$$

See Fig. 11 for values for arches of constant cross section.

Moments due to Unit Horizontal Displacement without Rotation of Ends

The loads on the analogous column section are now the following:

$P = 0$, $M_x = 1$, and $M_y = 0$.

From Eq. 1, the moments at both ends of each arch are

$$f = 0 + (1)\ (0.7704\ h)\ \frac{21.46}{h^2}\ \frac{EI_a}{L_a} = \frac{16.53}{h}\ \frac{EI_a}{L_a}$$

Values for arches of constant cross section are given in Fig. 11.

Thrust	Moments		Thrust
$+\frac{0.8427}{h}$	-1.0000	+0.4444	$-\frac{0.8427}{h}$
	0	-0.4444	$+\frac{0.3742}{h}$
$-\frac{0.3742}{h}$	+0.1974	0	
	0	0	
$+\frac{0.4685}{h}$	-0.8026	0	$-\frac{0.4685}{h}$
$+\frac{0.5837}{h}$	-1.0000	0	$-\frac{0.5837}{h}$

Figure 10. Horizontal Thrust - Unit Moment at One End, Far End Free.

Arch Shape	h/L	Moment Stiffness	Moment carry-over factor	Thrust due to unit moment (other end free to rotate)	Moment due to unit horizontal displacement (no end rotation)
Parabolic	0.1	8.554 EI/L	0.3299	6.278 /L	71.424 EI/L^2
	0.2	7.538	0.3221	3.174	31.635
	0.3	6.461	0.3161	2.149	18.271
	0.4	5.532	0.3123	1.637	11.885
	0.5	4.786	0.3108	1.329	8.340
Circular	0.1	8.706	0.3425	6.266	73.233
	0.2	7.794	0.3565	3.138	33.179
	0.3	6.687	0.3826	2.101	19.427
	0.4	5.591	0.4148	1.584	12.527
	0.5	4.634	0.4504	1.273	8.557
Semi-elliptic	0.1	12.429	0.5737	5.935	116.096
	0.2	8.978	0.5161	3.014	41.035
	0.3	6.885	0.4815	2.047	20.878
	0.4	5.548	0.4619	1.564	12.686
	0.5	4.634	0.4504	1.273	8.557

Figure 11. Constants for Arches of Constant Cross Section.